Hierarchic Shape Description via Singularity and Multiscaling

T. L. Kunii
Computer Sci. and Eng. Lab.
The University of Aizu
Fukushima 965-80 Japan

A. G. Belyaev
Center for Mathematical Sci.
The University of Aizu
Fukushima 965-80 Japan

E. V. Anoshkina
Computer Sci. and Eng. Lab.
The University of Aizu
Fukushima 965-80 Japan

S. Takahashi
Dept. of Information Sci.
The University of Tokyo
Tokyo 113 Japan

R. Huang
Computer Sci. and Eng. Lab.
The University of Aizu
Fukushima 965-80 Japan

O. G. Okunev
Computer Sci. and Eng. Lab.
The University of Aizu
Fukushima 965-80 Japan

Abstract

We introduce a new concept of ridges, ravines and related structures (skeletons) associated with surfaces in three-dimensional space that generalizes the medial axis transformation approach. The concept is based on singularity theory and involves both local and global geometric properties of the surface; it is invariant with respect to translations and rotations of the surface. It leads to a method of hierarchic description of surfaces that yields new approaches to shape coding, rendering and design. The extraction of the features is based on differential geometry of surfaces with consequent segmentation via multiscale analysis. Fourier feature recognition, dental shape reconstruction and medical imagery are a partial list of applications.

1 Introduction

Recent achievements of visual geometry and topology [6, 9] made many nontrivial mathematical concepts understandable. This made it possible to connect associative connections between abstract notions of modern geometry and some natural objects and led to new approaches in surface coding and animation [14, 23, 14, 24, 9].

On the other hand, recent advances in pattern recognition, computer vision and computer tomography have inspired fresh interest in such ancient features of a smooth surface as its parabolic line, umbilics and others that have previously been known only to speculists. However, handling these surface features calls for nontrivial geometric ideas. Such visual and natural features of a surface as its ridge-ravine lines have only recently come to be understood, following the development of the catastrophe theory and the associated branch of mathematics known as singularity theory in the works of R. Thom, V. I. Arnold and others.

The first serious mention of ridge-ravine lines on a surface as a viewpoint invariant was done by A. Guillstrand [12] ninety years ago. These surface features turned out to be closely connected with such optical notions as caustics and caustic singularities (the intensity of light is greater near a caustic and still greater near its singularities). Guillstrand worked on the accommodation of the eye lens and created the necessary fourth-order differential geometry to explain the relevant optics. He was awarded the Nobel Prize for Physiology and Medicine in 1911 for this work.

Since the description of the ridge-ravine lines is based on advanced differential geometry, and they were entirely disregarded in textbooks on differential geometry, some of them were rediscovered several times by researchers in structural geology [22], face recognition [10], and interpretation of magnetic resonance scans of the surface of the brain [11, 25].

A short historic review and a detailed mathematical treatment concerning caustics, ridge-ravine lines, and related surface features can be found in a very recent and excellent book of L. R. Porteous [24].

We should also note here that the ridge-ravine lines are known among mathematicians working in Japanese industry as "the principal curvature ex.
tremous curves" [13].

Another trend in shape analysis originated from the famous Medial Axis Transformation (MAT) introduced by U. Bresler [6]. MAT is one of the earliest, and probably the most widely studied, systematic technique for global shape description and pattern recognition (see, for example, [26] on applications of MAT to pattern recognition and [19] on 3D medial axis geometry for the nonsingular points of MAT skeletons). Nowadays 2D Medial Axis Transformation (2D-MAT) is also successfully applied in medical imagery [25, 18].

In this paper we study correlations between local and global properties of surfaces. Using the discovered relationships between the singularity of the distance function from a surface and singularities of the caustic of the normals to the surface, we find a connection between 3D MAT and the mesh of ridge-ravine lines in the surface, and suggest an effective procedure for ridge and ravine recognition [1, 2].

Our basic idea [1, 2] is to define ridge and ravine lines in a purely geometric manner, so that no particular coordinate system is involved in the definition. The new definition, designed in a form useful for animation, arose after the paper [15] where extracting geometric features is based on singularity theory and nets of primitives are used for garment wrinkling modeling and animation. Following the novel achievements of singularity theory [4, 3, 4] and visual geometry [8, 9], we developed the MAT approach and found the relationships between medial axis geometry (global surface features) and caustic singularities and ridge-ravine lines (local surface features). Roughly speaking, end points of the medial axis skeleton for a given surface are situated at singularities of the caustics and correspond to ridge-ravine lines in the surface. These investigations led us to new results in one of classical areas of mathematics, differential geometry of curves in a surface.

2 Ridges, Ravines and Distance Function Singularities

Let us consider a piecewise-smooth oriented hypersurface $M$ in the Euclidean space $\mathbb{R}^d$. Let $d$ be the distance function from a point $x$ of the Euclidean space to $M$ by

$$\text{dist}(x, M) = \inf_{y \in M} |x - y|^2.$$ 

The distance function to a piecewise-smooth hypersurface is continuous, but not necessarily smooth, even if the initial hypersurface is smooth. Let us consider the set of all singular points of the distance function to the hypersurface. It turns out that this set of singular points is a curve in a polyhedron (hence admitting a CW-complex structure), whose dimension in general position is equal to $d - 1$. We call this polyhedron the $R$-skeleton for $M$.

**Definition 2.1.** The $R$-skeleton of a given hypersurface $M$ is the set of all singular points of the mapping $\text{dist}(\cdot, M) : \mathbb{R}^d \to \mathbb{R}$.

The following basic assertion is easy, but we formulate it here as a theorem because of its significance.

**Theorem 2.1.** The $R$-skeleton is the closure of the set of all points that have at least two nearest neighbors.

Our concept of ridge-skeleton has some resemblance to the concept of spine [9].

Let us consider a closed hypersurface $M$ bounding a figure $F$. The medial axis transform (MAT) skeleton is defined as the locus of the centers of all maximum disks of $F$, that is, the disks contained in $F$ that are not contained in any other disk lying in $F$. Thus, the $R$-skeleton concept is a generalization of the MAT skeleton concept.

In order to distinguish ridges and ravines, we introduce the notions of a ridge skeleton and a ravine skeleton [2].

Let $M$ be endowed with an orientation $\mathbf{n}(p)$, where $\mathbf{n}(p)$ is the unit normal vector at the point $p$ of $M$.

**Definition 2.2.** The ridge skeleton of a given hypersurface $M$ is the connected component of the $R$-skeleton pointed at by the normal $\mathbf{n}(p)$. The ravine skeleton of a given hypersurface $M$ is the connected component of the $R$-skeleton pointed at by the normal $-\mathbf{n}(p)$.

Of course, in some cases there is only one connected component of the $R$-skeleton, which is either a ridge skeleton or a ravine skeleton, depending on the choice of orientation. If $M$ is a closed manifold, then the hypersurface $M$ separates $\mathbb{R}^d$ into two parts, and we can say that the ridge skeleton is the ridge skeleton for the complementary part.

A point $p$ of a polyhedron $P$ is called regular if $P$ is a manifold at $p$; that is, if there exists a neighborhood of $p$ in $P$ homeomorphic to a $(d - 1)$-dimensional Euclidean ball. We call a point $p$ of a polyhedron $P$
singular if \( P \) is not a manifold at \( p \); a singular point is interior if there is a set in \( P \) that contains \( p \) and is homeomorphic to a \((d - 1)\)-dimensional Euclidean ball; otherwise, the singular point is called a boundary.

The following theorem recently proved by I. A. Bogdanskii [7] describes the singular points of the \( R \)-skelton for a surface \( M \) in a general position.

**Theorem 2.2** A neighborhood of a singular point of the \( R \)-skelton for a surface \( M \) in a general position, can be homeomorphic to one of the following polyhedra: a half-disk, a disk with a glued half disk, a disk with a glued quarter disk, and a disk with three glued quarter-disks.

Below we concentrate our attention on the boundary singular points of the \( R \)-skelton.

**Definition 2.3** Pseudoridges and pseudovoronoï are the connected components of the boundary singular points of the ridge skeleton and the voronoï-skeleton respectively.

**Definition 2.4** A ridge (ravin) of a piecewise smooth hypersurface \( M \) in the Euclidean space \( \mathbb{R}^d \) is the set of all points of \( M \) that realize the infimum of the distance function at the points in the pseudoridge (pseudovoronoï).

Fig. 2 demonstrates the relation between boundary singular points of the \( R \)-skelton and the ridge (ravin) lines; we also show here possible location of neighborhoods of the typical points of the \( R \)-skelton is also shown.

![Figure 1: R-skeleton and relative objects](image)

**3 Caustics: General Theory**

Unfortunately, Definition 3.1 is very complicated from the practical computational point of view.

Therefore, we need to find another way to recognize the ravinage and ravines on a given surface.

It turns out that the concepts of a ridge and a caustics are closely related to the concept of \( C \)-structure. Let us recall some properties of this object, which play an important role in optics. Precisely, in our search through available literature we could not find a simple introduction to the theory of caustics, so we derive some basic formulas and prove some basic properties of the caustic here. Of course, a mathematician can get much more from the monograph [4]. Below we only consider caustics generated by \( M \) to a given hypersurface \( M \). In fact, we only deal with surfaces in a three-dimensional space and curves in a two-dimensional plane, but for non-geometric applications it is sometimes also useful to consider multi-dimensional caustics; therefore, some calculations are performed for a hypersurface in \( \mathbb{R}^d \).

Let \( M \) be a smooth hypersurface with local coordinates \((x_1, \ldots, x_{d-1})\) in \( \mathbb{R}^d \). In a sufficiently small neighborhood of \( M \) it is possible to introduce a local coordinate system \((x_1, \ldots, x_{d-1}, t)\) so that \( t \) is the distance from the surface along the normal, and \((x_1, \ldots, x_{d-1}, t)\) are the coordinates of the base point of the normal at the surface. Consider the Jacobian \( J(x_1, \ldots, x_{d-1}, t) = \frac{\partial(x_1, \ldots, x_{d-1}, t)}{\partial(x_1, \ldots, x_{d-1}, t)} \) of Cartesian coordinates in \( \mathbb{R}^d \). A point \( x = x(t, \ldots, t) \) is called focal if \( J(x_1, \ldots, t) = 0 \) at \( x \).

**Definition 3.1** The set of all focal points is called the caustic of the surface.

Sometimes below we will use subscripts to denote partial derivatives, and for brevity perform calculations for the three-dimensional case (although they may be easily generalized to arbitrary dimension).

Let \( M \) be a hypersurface in \( \mathbb{R}^d \) defined by the radius vector \( r = r(u, v) \) and let \( n = n(p) \) be the unit normal vector to \( M \) at a point \( p \in M \). We always can choose local coordinates \( u, v \) so that \((r_1, r_2, M) = (u, v, n_1, n_2, M)\) is an orthonormal base at the point \( p \) and the tangent vectors \( r_1, r_2 \) coincide with the principal directions corresponding to the principal curvature radii \( R_1 \) and \( R_2 \). Therefore, \( r_1 = \frac{1}{R_1} \) and \( r_2 = \frac{1}{R_2} \), and the Frenet-Serret formulas. The set of all focal points satisfies the equation

\[
J(u, v, t) = \det \begin{pmatrix}
\frac{\partial^2 (r + t n)}{\partial u^2} & \frac{\partial (r + t n)}{\partial u} & \frac{\partial (r + t n)}{\partial t} \\
\frac{\partial (r + t n)}{\partial u} & \frac{\partial^2 (r + t n)}{\partial v^2} & \frac{\partial (r + t n)}{\partial v} \\
\frac{\partial (r + t n)}{\partial t} & \frac{\partial (r + t n)}{\partial v} & \frac{\partial^2 (r + t n)}{\partial t^2}
\end{pmatrix} = 0.
\]
where \( a, b, c \) is the mixed product of vectors \( a, b \) and \( c \). Thus, the roots of this equation coincide with the principal curvature radii \( R_1 = 1/\kappa_1, R_2 = 1/\kappa_2 \), and we come to

Theorem 3.3 The caustic is the surface of the principal centers of the curvatures of \( M \).

One can see that there are therefore two caustics for a surface in a three-dimensional space. Let us now prove that the family of normals to \( M \) is tangent to the caustics at principal centers of the curvatures of \( M \). For a given smooth surface \( M \), the equations of the curvatures are

\[
\tau = \tau(u, v) + R_
u(u, v) \cdot h(u, v), \quad i = 1, 2.
\]

Where, normals to the tangent planes to the caustics \( m_i \), can be calculated as

\[
m_i = \left( n \times \frac{\partial R_i}{\partial u} \cdot n \right) \left( n \times \frac{\partial R_i}{\partial v} \cdot n \right) = n \times \left( = \kappa_2 \nu_2 \right), \quad i = 1, 2,
\]

where \( n \times h \) is the cross product of vectors \( n \) and \( h \).

Using the Penot–Serret formulas, we get

\[
\begin{align*}
\tau_i &= x_i - n_i - \frac{\partial n}{\partial u} \cdot x_i - \frac{\partial n}{\partial v} \cdot x_i, \\
\kappa_i &= \nabla x_i - \nabla \cdot n_i = n_i \times \left( \frac{\partial R_i}{\partial u} \right) = \kappa_i h_i, \\
\nu_i &= n_i \times \left( \frac{\partial R_i}{\partial v} \right) = \kappa_i \nu_i, \\
\end{align*}
\]

Therefore,

\[
m_i = \left( 1 - \frac{\partial R_i}{\partial u} \right) \left( R_i - 1 \right) \frac{\partial R_i}{\partial v} + \frac{\partial R_i}{\partial u} \left( R_i - 1 \right) \frac{\partial R_i}{\partial v} - \frac{\partial R_i}{\partial u} \left( R_i - 1 \right) \frac{\partial R_i}{\partial v}.
\]

Since the factor at the normal vector \( n \) is equal to zero, and the tangent vectors \( e_0, e_2 \), are orthogonal to \( n \), the projection of the tangent point to the caustic along the normal vector \( n \) is on the tangent plane to \( M \) is a line. Therefore, the following proposition holds.

Theorem 3.2 The caustics is an envelope of the family of all normals to \( M \). The caustics is tangent to any normal to \( M \) at the principal order of the curvature of \( M \).

Let us now recall some basic notions from the Morse theory [17]. A point \( m \) of a manifold \( M \) is called a critical point of a function \( f \) defined in a neighborhood of \( m \) if all partial derivatives of \( f \) vanish at \( m \). At a critical point, the function \( f \) is thus approximated by a quadratic form whose coefficients are the second order partial derivatives. The matrix of this quadratic form is called the Hessian matrix of the function \( f \) at \( m \).

A critical point \( m \) is called a Morse–critical point if the Hessian is nondegenerate at \( m \).

Let us consider a family of distance functions (defined on \( M \)) from a point (considered as a parameter of the family) to a given surface \( M \).

The point \( m \) is critical (hence we fix the parameter of the family of functions defined on \( M \)). It is easy to see that the distance function has an extreme value at the point of \( M \) that corresponds to the chosen point of the caustic. Moreover, by the previous theorem, the Hessian of the distance function degenerates at this point. Thus, we get (see also [4]).

Theorem 3.3 The caustics is formed by the parameter (points) for which the distance function has non Morse critical points.

Let us now look at the caustics from the optical point of view. Recall that a light wavefront generated by \( M \) is the surface which is the set of endpoints of the normal vectors to the surface of a fixed length. The wavefronts that are near the original vertices are smooth, but as the length increases, as some moment singularities may arise. The curvature radii of a wavefront at the distance \( t \) from \( M \) are \( R_1 = t \) and \( R_2 = t \). This gives

Theorem 4.4 The caustics is swept by the congruities of moving wavefronts.

It also follows from this theorem that the caustics are the back light rays confined by a given surface \( M \) in the normal direction are concentrated. At the singular points of the caustics the concentration of the light rays is greater.

4 Caustics for Ridge and Ravine Recognition

Let us now apply caustics to ridge and ravine recognition. Consider a point in the Ridgeset. The distance function from this point to a point \( M \) generally has two equal minima. Let us move this point to a pseudocurve (pseudocurve). When the point achieves a pseudocurve (pseudocurve), two equal minima merge. It follows that a point in a pseudocurve (pseudocurve) corresponds to the distance function

[34]
having a non-Morse critical point, namely, a degenerate minimum. Therefore, all pseudoridges (pseudoravines) lie in caustics. Moreover, since the distance function has a degenerate minimum, then a point on a pseudoridge or pseudoravine (where two equal minima merge) belongs to set of the singularities of the caustics. Fig. 2 demonstrates these arguments.

We come to

**Theorem 4.1.** Pseudoridges and pseudoravines lie at caustic singularities.

![Diagram of caustic and pseudoridge](image)

**Figure 2:** Zone of distance functions

This theorem gives us a key for explicit calculation of ridges and ravines.

Let us find the singular points of the caustics. For a given smooth surface $M$, the equations of the caustics are

$$ r = H(u, v) + R(u, v) \cdot n(u, v), \quad i = 1, 2. $$

The equations for the singular points of the caustics have the form

$$ \left( r + \frac{\partial R}{\partial u} n_i \right) \times \left( r + \frac{\partial R}{\partial v} n_i \right) = 0, \quad i = 1, 2. $$

Again choose the coordinate system associated with the curvature lines and using the Frenet-Serret formalism, we obtain the equations for the singular points of the caustics in the form

$$ \mathbf{n} \left( 1 - \frac{R_i^2}{R_i} + \frac{R_i^2}{R_i R_j} \right) + \mathbf{r} \cdot \frac{\partial R_j}{\partial u} (R_i - 1) = 0. $$

Obviously, the first term is equal to zero. Since $r_i$ and $r_j$ are linearly independent, we obtain three equations, each of which determines points that correspond to singular points of the caustics:

$$ R_i = R_i, \quad \frac{\partial R_i}{\partial u} = 0, \quad \frac{\partial R_i}{\partial v} = 0. \quad (1) $$

Thus we have proved

**Theorem 4.2.** The singularities of the caustics correspond to the points of $M$ where the principal curvatures are equal (umbilic points) or the principal curvature has an extremum along the corresponding principal direction.

Let us note that not all these points lie in ridges or ravines. Actually, for an ellipse in a plane as in Fig. 3,

$$ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b, $$

the caustic (the evolute, or, in other words, the locus of the centers of curvature) is a set

$$ \{ax^{2/3} + (by)^{2/3} = (a^2 - b^2)^{1/3} \} $$

with four semicircular cusps

$$ \{b \left( 1 - \frac{b^2}{a^2} \right), 0 \} \text{ and } \{0, \pm b \left( 1 - \frac{a^2}{b^2} \right) \}. $$

But the ridge-skeleton is $\{y = 0, |x| < a(1 - b^2/a^2)\}$, the points $\{a \left( 1 - b^2/a^2 \right), 0 \}$ are the pseudoridges corresponding to the ridges $(2a, 0)$, and two other extreme points $\{0, \pm b \left( 1 - c^2 \right) / b^2 \}$ do not correspond to any ridge or revine, as it is shown in Fig. 3.

Nevertheless, the situation on a plane is clear. Let us consider a simple closed curve $M$ in the plane. Let $M$ be oriented counterclockwise and be given by a vector function $r = r(a)$ where $a$ is the arclength parameterization of $M$. The curvature of $M$ at a point $a$ is

$$ \kappa(a) = \frac{d^2r(a)}{da^2} \cdot a(a), $$

where $a \cdot b$ is the scalar product of vectors $a$ and $b$.

**Theorem 4.3.** The function $\kappa(a)$ has a local positive maximum at a point if the point is a ridge, and $\kappa(a)$ has a local negative minimum at a point if the point is a revine.
Figure 2: Example with an ellipse

Figure 4: General situation in a plane

We also consider point surface features which lie in ridge-valley lines and can be used to characterize maps.

Definition 4.2 An extreme ridge point is a point where $k_{max}$ has a local positive maximum. An extreme ravine point is a point where $k_{min}$ has a local negative minimum.

A ridge and ravine extraction procedure may now be described as follows. We go through the points of $M$, calculating principal curvatures and the corresponding principal directions.

1. Calculate the first and the second fundamental forms at the point:
   \[ I = E(u,v)du^2 + 2F(u,v)dudv + G(u,v)dvd^2, \]
   \[ II = E(u,v)du^2 + 2F(u,v)dudv + N(u,v)dvd^2, \]
   \[ E = r_m, \quad F = r_m r_n, \quad G = r_n^{-1}, \]
   \[ r_m = r_m, \quad M = e_m, \quad N = r_n - n. \]

2. Find the Gaussian, mean and principal curvatures:
   \[ K_g = \frac{EG - F^2}{EG}, \quad K_m = \frac{1}{2} \frac{E(G'' - 2F'')}{E(G - F^2)}, \]
   \[ K_{max} = K_m + \sqrt{K_m^2 - K_g}, \quad K_{min} = K_m - \sqrt{K_m^2 - K_g}. \]

3. Find the principal directions $v_{max}$ and $v_{min}$ as the eigenvectors corresponding to the eigenvalues $k_{max}$ and $k_{min}$.

The situation in the three-dimensional case is more complicated. We calculated the critical points of the covariance and the points on the surface that correspond to the singularities of the covariance. Those points are the critical points of the covariance (where the principal curvatures are equal) and the points where the derivatives of the principal curvatures along the respective principal directions are equal to zero (see (1)). But not all those points correspond to ridges and ravines (see the example with an ellipse).

Let $k_{max}$ and $k_{min}$ be the principal curvatures ($k_{max} > k_{min}$), and $\lambda_{max}$ and $\lambda_{min}$ the corresponding principal directions. Consider also the Gaussian curvature $E = \lambda_{max} \lambda_{min}$ and the mean curvature $K_m = \frac{1}{2}(k_{max} + k_{min})$. Note that if $K_m$ is positive in some region on $M$, then the curves generated by the normals at this region lie on the same side of $M$. If $K_m$ is negative, then the curves lie at different sides of $M$. If $K_m = 0$, then the normals of the region form only one cusp (the other "moves to infinity"). By analogy with two-dimensional case, we give

Definition 4.1 A ridge is a set of points where $k_{max}$ has a local positive maximum along $\tau_{max}$. A ravine is a set of points where $k_{min}$ has a local negative minimum along $\tau_{min}$. Taking into account theorems 4.1, 4.2, 4.3 and the argument so above, we imply the equivalence between this definition of ridge and ravine curves and the previous one. Nevertheless, the question of choice of a proper definition still remains open.
and $b_n$ respectively from the spectral problem

$$\begin{bmatrix}
E_k - L & F_k - M \\
F_k - M & G_k - N
\end{bmatrix} = 0$$

and go to the next point. Thus, we get the data as a quadruple

$$(v_{\text{max}}(p), v_{\text{min}}(p), k_{\text{max}}(p), k_{\text{min}}(p))$$

where a point $p$ ranges over $M$. Tracing each line of the curvatures (integral curves of the principal directions $v_{\text{max}}$ and $v_{\text{min}}$), we find points of a ridge or a ravine in accordance with Definition 4.1.

5 Shape Coding, Restoration and Rendering

The ridge-skeletons and ravine-skeletons introduced as above may be used for coding surfaces in the following manner. Suppose we code a surface $M$ by the $R$-skeleton $R$, with the function $r$ representing the distance; thus $r$ is the radius of the sphere with the center at the given point of $R$, tangent to $M$ at two points. Thus, the surface $M$ must be tangent to each sphere in the family of spheres with centers in $R$ of radius $r$. More precisely, suppose the surface $M$ is given by parametric equation $x = x(u,v)$, and the radius at $x(u,v)$ is $r(u,v)$. The surface $M$ must be tangent to the sphere of radius $r(u,v)$ with the center $x(u,v)$. If we denote by $m(u,v)$ one of two points where the sphere and $M$ are tangent and $q(u,v) = m(u,v) - x(u,v)$, then we get the system of equations

$$\begin{align}
(q, q) &= r^2, \\
q_v &= \frac{\partial r}{\partial v}, \\
q_u &= \frac{\partial r}{\partial u},
\end{align}$$

where $(\cdot , \cdot)$ is the scalar product and $q$, the radius of the sphere, is orthogonal to the vectors $\frac{\partial m}{\partial v}$, $\frac{\partial m}{\partial u}$, tangent to the surface $M$.

Substituting $m = x + q$, we get

$$\begin{align}
(q, q) &= r^2, \\
q_v &= \frac{\partial r}{\partial v}, \\
q_u &= \frac{\partial r}{\partial u},
\end{align}$$

Differentiating the first equation with respect to $u$ and $v$, we get $q_u = \frac{\partial r}{\partial u}$ and $q_v = \frac{\partial r}{\partial v}$, so

the above system is equivalent to

$$\begin{align}
(q, q) &= r^2, \\
q_v &= -\frac{\partial r}{\partial v}, \\
q_u &= -\frac{\partial r}{\partial u}.
\end{align}$$

This is a system of one quadratic and two linear algebraic equations with respect to $q = (q^1, q^2, q^3)$, and $R$ is easily solvable; the solutions exist if and only if the sum of the derivatives of $r$ in any direction (in the parametric plane $(u,v)$) does not exceed the absolute value of the derivative of $s$ in the same direction.

It may seem that we gain nothing if we use this way of coding surfaces. Indeed, now we need to keep four functions of two variables, $r^i(u,v), i = 1, 2, 3$, and $r(u,v)$, while in the parametric representation of the surface we need only three. However, in many applications it is important to keep the shape of the surface more precisely near the ridges and ravines, and less precisely in the "regular areas". In our model, means that we keep with higher accuracy the functions $x(u,v)$ and $r(u,v)$ near $R$ - an object of dimension one, and in the rest of $R$ is shape and the values of $r$ may be kept with less accuracy; in the extreme case, the values of $x(u,v)$ and $r(u,v)$ can be simply interpolated far from their known values on a pseudocircle (pseudoravine). Furthermore, in some problems of shape modeling, such as crown shape generation in dental CAD systems, it is useful to have a tool for construction of a surface with prescribed positions of ridges and ravines; the above approach reduces the problem of moving ridges and ravines in the surfaces to moving boundaries of the $R$-skeleton.

6 Hierarchic Shape Description

We can classify surface features as follows:

- Global features, which describe global surface geometry and topology. Among them we should mention the Euler characteristic [27], the Reeb graph [23, 24], the $R$-skeleton and other.
- Local features, which are determined by local properties of the surface. We may further classify them into

1. Surface features, such as mean and Gaussian curvatures.

2. Line features. A wide variety of such features can be found, for example, in [12]. Here we add the ridge and ravine lines introduced as above.

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3. Point features. Among these, we distinguish pits, passes, peaks (if we have a distinguished direction and a height function along it), umbilicals, extreme ridge and ravine points (see Definition 4.2).

Thus, our approach lead us to the natural hierarchical classification: R-skeleton, ridge and ravine skeletons → ridge and ravine lines → extreme ridge and ravine points.

7 Multiscale Analysis

Multiscale Analysis of surfaces allows a hierarchical representation of their composing features. To represent a surface at a given scale, different from the one in which it was given originally, structures that are insignificant at that scale have to be eliminated. We perform multiscaling in two steps:

- The choice of the degree of smoothness.
- Thresholding.

A surface is smoothed by the convolution with a smoothing function which is dilated by a scale factor. An smoothing function may be either Gaussian or a properly chosen wavelet.

- Thresholding.
  1. Remove from the ridges (ravines) curve the points where the absolute value of the principal curvature is smaller than a given threshold.
  2. Remove all ridge (ravine) curves whose length is smaller than a given threshold.
  3. Remove all ridge (ravine) curves along which the average absolute value of the relevant principal curvature is smaller than a given threshold.

8 Experimental Results

A ridge and ravine extraction algorithm has been developed and implemented on an IRIS Oxyx workstation. A display of 3D surfaces with the extracted ridges and ravines was obtained.

Fig. 5 shows six resulting images, in which purple points are ravines and cyan points are ridges. The following surfaces are shown:

- The top two images show random 3D surfaces;
- The two images in the middle and the right image at the bottom show 3D polynomial-trigonometric surfaces;
- The left image at the bottom exposes a 3D polynomial surface.

9 Conclusions and Problems for Future Research

Surface degeneracies, such as ridges, ravines and cusps have important applications in animation, computer simulation, geometric design, and pattern recognition. In this paper we have defined ridges and ravines in a purely geometric manner, so that no particular coordinate system is involved in the definition. The method leads to definition of associated geometric structures (skeletons, pitchforks, and pseudo-ravines) that can be used for coding and animating modeled surfaces. Furthermore, we define the features of the surface in terms of singularities of functions associated with the surface, which makes the results of singularity theory applicable to the study of the features.

Although there is still a lot of work to be done with the above approach, both in refining the mathematical model and development of effective algorithms for extraction of ridges, ravines, and skeletons, the results here are extremely promising.

One possible direction for a further research is application to terrain feature recognition. In the case of mountains shapes, we have a distinguished direction, and we may use the coordinate along this direction as the height function. The Reeb graph is a very useful tool for mountains shape coding [23]. The ridge-skeleton introduced above has no options for coding passes and peaks, so some modifications of our ridge-skeleton and ravine-skeleton concepts could be useful in this case. Let us consider all cross-sections of a given surface perpendicular to the distinguished direction and a given (Morse) height function. For each cross-section let us construct its ridge-skeleton and ravine-skeleton, find ridges and ravines, and place them together (thus, we come to a new definition of ridges and ravines, which is a revision of the ones introduced in [13] and [17]). The resulting polyhedron can be used for coding the ridges, passes, peaks and their disposition.

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Figure 8: Ridges and ravines on random and polynomial trigonometric surfaces
References


